EVERY FRAME IS A SUM OF THREE
(BUT NOT TWO) ORTHONORMAL BASES -
AND OTHER FRAME REPRESENTATIONS

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ABSTRACT. We show that every frame for a Hilbert space H can be written as a (multiple of a) sum of three orthonormal bases for H. We next show that this result is best possible by including a result of N.J. Kalton: A frame can be represented as a linear combination of two orthonormal bases if and only if it is a Riesz basis. We further show that every frame can be written as a (multiple of a) sum of two tight frames with frame bounds one or a sum of an orthonormal basis and a Riesz basis for H. Finally, every frame can be written as a (multiple of a) average of two orthonormal bases for a larger Hilbert space.

1.Frames as Operators

If H is a Hilbert space, we denote the set of all bounded operators $T : H \to H$ by $B(H)$. We will always use (e_n) to denote an orthonormal basis on H. Recall that a sequence (x_n) in a Hilbert space H is called a **frame** for H if there are constants $0 < A \le B$ so that for all $x \in H$ we have

$$A\|x\|^2 \le \sum_n | < x, x_n > |^2 \le B\|x\|^2.$$

We call A, B the **frame bounds** for the frame and if $A = B$, we call this a **tight frame**. The frame definition has many equivalent forms. We will work here with frames thought of as operators on H. That is, a sequence (x_n) is a frame on H if and only if there is an operator $T : H \to H$ so that $Te_n = x_n$ and T is an onto

map. This equivalence is easily checked. For one direction, given the operator T we just check that

$$T^*(x) = \sum_n <x, x_n> e_n.$$

Since T is onto, T^* is an (into) isomorphism. Hence, TT^* is an onto isomorphism and

$$TT^*(x) = \sum_n <x, x_n> x_n,$$

is the so called **frame operator**. Conversely, if (x_n) is a frame, then (x_n) is a Hilbertian sequence in H. That is, for all sequences of scalars (a_n) we have

$$\sum_n a_n x_n$$

converges in H. Hence, if we define $T : H \to H$ by $Te_n = x_n$, then T is a bounded linear operator from H to H. Also, TT^* is the frame operator for this frame and hence TT^* is an onto isomorphism. Therefore, T must be an onto map.

So we can consider the "equivalence" between frames and onto maps on H. That is, if we have a theorem about onto maps on H (or about all bounded operators on H), then we have a theorem about frames.

2. Frames as Sums

All Hilbert spaces will be complex (at the end we will discuss what happens in the real case). The results we use from Operator Theory can be found in any standard book in Functional Analysis [2], or books on Hilbert space theory [1,3]. Recall that a **unitary operator** $U : H \to H$ is an onto isometry, a **partial isometry** is an operator which is an isometry on the orthogonal complement of its kernel, a **co-isometry** is an operator whose adjoint is an into isometry, and a **maximal partial isometry** is either an isometry or a co-isometry. An operator $T \in B(H)$ is called a **positive operator** if for all $x \in H$ we have $<Tx, x> \geq 0$. That is, $<Tx, x>$ is both a real number and positive. The main result we will use is the fact that every $T \in B(H)$ has a representation in the form $T = VP$ (called the**Polar decomposition of T**) where V is a maximal partial isometry and P is a positive operator. Moreover, we may assume that ker V = ker P. Also, every positive operator P on H with $\|P\| \leq 1$ can be written in the form $P = \frac{1}{2}(W+W^*)$, where $W = P + i\sqrt{1 - P^2}$ is unitary.

Our first result is well known in operator theory, but since it is never formally stated, we include a sketch of the proof.

Proposition 2.1. *If $T : H \to H$ is a bounded linear operator, then $T = a(U_1 + U_2 + U_3)$, where each U_j is a unitary operator and for any $0 < \epsilon < 1$, we can specify $a = \frac{\|T\|}{1-\epsilon}$.*

Proof. Fix $0 < \epsilon < 1$ and let

$$S = \frac{1}{2}I + \frac{1-\epsilon}{2}\frac{T}{\|T\|}.$$

Then

$$\|I - S\| = \|\frac{1}{2}I - \frac{1-\epsilon}{2}\frac{T}{\|T\|}\| \le \frac{1}{2} + \frac{1-\epsilon}{2} < 1.$$

It follows that S is an onto isomorphism. We now write the polar decomposition of S as $S = VP$ where V is a maximal partial isometry and P is a positive operator and $\ker V = \ker P$. Since S is an isomorphism, P is an isomorphism and V is a unitary operator (Since a necessary and sufficient condition that V be an isometry is that S be 1-1, and a necessary and sufficient condition that V be a co-isometry is that S has dense range [3]). Also, $\|S\| < 1$ implies that $\|P\| \le 1$, and hence $P = \frac{1}{2}(W + W^*)$. Now we have the representation,

$$S = \frac{1}{2}(VW + VW^*),$$

where VW, VW^* are unitary. Finally,

$$T = \frac{\|T\|}{1 - \epsilon}(VW + VW^* - I)$$

is the required decomposition of T.

As an immediate consequence of Proposition 2.1 we have,

Corollary 2.2. *If $(x_i)_{i \in I}$ is a frame for a Hilbert space H with upper frame bound B, then for every $\epsilon > 0$, there are orthonormal bases (f_i), (g_i), (h_i) for H and a constant $a = B(1 + \epsilon)$ so that*

$$x_i = a(f_i + g_i + h_i), \quad \forall\, i \in I.$$

Proof. If we choose an orthonormal basis (e_i) for H, then the operator $T : H \to H$ given by $T(e_i) = x_i$ is an onto map and $\|T\| = B$. By Proposition 2.1, we can write $T = a(U_1 + U_2 + U_3)$ where each U_j, $1 \le j \le 3$ is a unitary operator. It follows that $(U_j e_i)_{i \in I}$ is an orthonormal basis for H for $1 \le j \le 3$ and by the proof of Proposition 2.1 (with $\frac{1}{1-\epsilon}$ there equaling $1 + \epsilon$ here)we get the estimate for a.

Corollary 2.2 cannot be improved to represent the frame as a sum of only two orthonormal bases (or even orthonormal sequences) in general, as the next example shows. We call a frame a **normalized tight frame** if it is a tight frame with frame bounds 1. Even though the proposition following the example supersedes it, we have included it here since it works in both the real and complex case and clearly illuminates the difficulties encountered in such a representation.

Example 2.3. *There is a normalized tight frame for a (real or complex) Hilbert space H which cannot be written as any linear combination of two orthonormal sequences in H.*

Proof. Let (e_i) be an orthonormal basis for H and consider the normalized tight frame: $x_1 = 0$, and for all $1 \leq i$, $x_{i+1} = e_i$. We proceed by way of contradiction. If we can find orthonormal sequences (f_i), (g_i) in H and numbers a, b so that $x_i = af_i + bg_i$, for all $i \in I$ then

$$x_1 = 0 = af_1 + bg_1.$$

Hence, if $a \neq 0 \neq b$, then span $f_1 =$ span g_1 and orthogonality imply

$$\text{span}(f_i)_{i=2}^{\infty} = \text{span}(g_i)_{i=2}^{\infty} \neq H,$$

while

$$\text{span}(af_i + bg_i)_{i=2}^{\infty} = \text{span}(e_i)_{i=1}^{\infty} = H.$$

This contradiction completes the proof of the Example.

Note that the above example does not even allow us to find two sequences in H which are only orthonormal bases for their spans and add up to our frame.

The following result, due to N.J. Kalton and appearing here with his permission, gives a complete characterization of those frames which can be written as linear combinations of two orthonormal bases as precisely the class of Riesz bases. Again, it comes from a result in operator theory.

Proposition 2.4. *If $T \in B(H)$ is onto, then T can be written as a linear combination of two unitary operators if and only if T is invertible.*

Proof. $\Leftarrow$: If T is invertible, then by the proof of Proposition 2.1, we have that $T = a(U_1 + U_2)$, where U_1, U_2 are unitary operators.

$\Rightarrow$: Suppose $T = aU_1 + bU_2$, where U_1, U_2 are unitary operators. If either of a or b equals zero, we are done. So without loss of generality, (after dividing by the

smaller of the two and observing that T is invertible if and only if $\frac{1}{a}T$ is invertible) we may assume that

$$T = U_1 + aU_2,$$

where $|a| \geq 1$. For all $0 \leq t < 1/2$, let

$$S_t = tU_1 + (1 - t)aU_2.$$

We observe that for all $0 \leq t < 1/2$, the operator S_t is a (possibly into) isomorphism. To see this we calculate for all $f \in H$,

$$\|S_t(f)\| \geq (1 - t)|a|\|U_2(f)\| - t\|U_1(f)\| = [(1 - t)|a| - t]\|f\|.$$

Since $0 \leq t < 1/2$ and $|a| \geq 1$, we have that $[(1 - t)|a| - t] > 0$ and so S_t is an isomorphism. Also, by our assumption that T is onto, we have that $S_{1/2}$ is onto. A result of Kalton, Peck, and Roberts [5], Proposition 7.8 (this is done for open maps but works perfectly well for onto maps) we have that the onto maps for an open set in $B(H)$. Since

$$\lim_{t \to 1/2} S_t = \frac{1}{2}T,$$

it follows that for t close enough to $1/2$, the operator S_t is onto. Hence S_t is invertible (being an isomorphism). But Proposition 7.9 of [5] says that the invertible operators forms a clopen (both closed and open) subset of the onto maps in $B(H)$. Therefore, $S_{1/2}$ is also an invertible operator. Hence, $T = 2S_{1/2}$ is invertible.

The corresponding result for frames looks like:

Proposition 2.5. *A frame (f_i) for H can be written as a linear combination of two orthonormal bases for H if and only if (f_i) is a Riesz basis for H.*

Although an arbitrary frame cannot be written as a multiple of a sum of two orthonormal sequences in the space, it can be written as a multiple of a sum of two normalized tight frames in the space. This follows from another general result from operator theory concerning the decomposition of an operator. That is, every operator T on a Hilbert space can be written in the form

$$(2.1) \qquad\qquad T = VP = \frac{\|T\|V}{2}(W + W^*),$$

where W is unitary and V is a maximal partial isometry. It follows that VW and VW^* are maximal partial isometries. That is, each of these operators is either an isometry or a co-isometry. However, if T induces a frame on H then T has dense range and so V must be a co-isometry.

Proposition 2.6. *If T is a co-isometry on H, and if (e_i) is a orthonormal basis for H, then (Te_i) is a normalized tight frame for H.*

Proof. Since T is a co-isometry, T^* is an isometry. Hence, for all $f \in H$,

$$\|f\|^2 = \|T^* f\|^2 = \sum_i | < T^* f, e_i > |^2 = \sum_i | < f, Te_i > |^2.$$

Therefore, (Te_i) is a normalized tight frame for H.

From our discussion preceeding Proposition 2.6, we see that any onto map $T : H \to H$ (inducing the frame (Te_i) on H) can be written in the form given in (2.1) where each of VW, VW^* is a co-isometry. This combined with Proposition 2.6 gives immediately

Proposition 2.7. *Every frame for a Hilbert space is (a multiple of) the sum of two normalized tight frames for H.*

Proposition 2.7 should be compared to the fact that every frame is equivalent to a normalized tight frame. As we saw earlier, even normalized tight frames may not be the sum of two orthonormal sequences in H. But, if we are willing to "expand" the Hilbert space, then we can get a good representation. The main ideas below are part of a nice alternate approach to frames due to Han and Larson [4].

Proposition 2.8. *If (x_i) is a normalized tight frame for H, then there is a Hilbert space $H \subset K$ and two orthonormal bases (f_i), (g_i) for K so that (x_i) is the average of (f_i) and (g_i).*

Proof. Let $Q : H \to H$ be the onto map $Qe_i = x_i$. Then (x_i) is a normalized tight frame implies that Q is a quotient map and hence Q^* is an isometry. By identifying x_i with $Q^* x_i$ and H with $Q^*(H)$ we can let $K = H$ and choose the orthogonal projection $P : K = H \to Q^*(H)$ and observe that $P(e_i) = Q^* x_i$. That is, for all i, j we have:

$$< Q^* x_i, Pe_j >=< PQ^* x_i, e_j >=< Q^* x_i, e_j >=< x_i, Qe_j >$$

$$=< x_i, x_j >=< Q^* x_i, Q^* x_j > .$$

But, $(Q^* x_i)$ spans $Q^* H$ and so $Pe_j = Q^* x_j$. Finally, $(Pe_i + (I - P)e_i)$ and $(Pe_i - (I - P)e_i)$ are both orthonormal bases for K and their average is $Pe_i = Q^* x_i$.

Since every frame is equivalent to a normalized tight frame we have:

Corollary 2.9. *Every frame is equivalent to a frame which is an average of two orthonormal bases for a larger Hilbert space.*

Riesz bases are a little more general than orthonormal bases, and so we get a stronger result relative to frames.

Proposition 2.10. *Every frame for a Hilbert space H is (a multiple of) the sum of a orthonormal basis for H and a Riesz basis for H.*

Proof. We proceed as in Proposition 2.1 with a slight change. Given a Pre-frame operator $T : H \to H$ with our frame being (Te_i), define an operator S by

$$S = \frac{3}{4}I + \frac{1}{4}(1 - \epsilon)\frac{T}{\|T\|}.$$

Then again we have $\|I - S\| < 1$ and $\|S\| \leq 1$, so S is an invertible operator and as in the proof of Proposition 2.1 we can write

$$S = \frac{1}{2}(W + W^*),$$

where W is a unitary operator. (Note that here W is taking the place of VW in Proposition 2.1). Now we have:

$$T = \frac{4\|T\|}{(1 - \epsilon)}\left[\frac{1}{2}(W + W^*) - \frac{3}{4}I\right].$$

Hence,

$$T = \frac{2\|T\|}{(1 - \epsilon)}[W + R], \quad \text{where} \quad R = W^* - \frac{3}{2}I.$$

Now, W is unitary so (We_i) is an orthonormal basis, and W^* is unitary implies that R is an isomorphism (possibly into). But, it is easily checked that R is onto since

$$\|I - \frac{-1}{2}R\| = \|\frac{1}{4}I + \frac{1}{2}W^*\| < 1.$$

Thus $\frac{-1}{2}R$ is an invertible operator and hence R is an invertible operator. Since R is an invertible operator, (Re_i) is a Riesz basis for H.

Remark 2.11. *Again, what we have really used in Proposition 2.10 is just a result from operator theory which says that every operator on a Hilbert space is a multiple of the sum of a unitary operator on H and an invertible operator on H.*

Remark 2.12. *In the real case, we cannot write a positive operator as an average of two unitaries. This result in the complex case comes from the representation of the extreme points of the ball of $B(H)$ and the fact that every positive operator P with $\|P\| = 1$ is actually an average of two extreme points. In the real case, we lose this representation but do have a similar one with a representation in terms of 16 operators. Thus, we can recapture the above results with "larger sums". For example, we can write a frame in a real Hilbert space as a multiple of a sum of 16 orthonormal bases or 4 Riesz bases, and a normalized tight frame can be written as the sum of four orthonormal bases etc.*

References

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